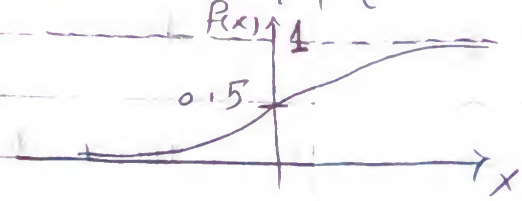
محاضرة $\boxed{3}$ — د. محمود فهمي — الأربعاء — 12/10/2016

Sigmoidal functions in neural networks

Sigmoidal functions

Binary Sigmoid

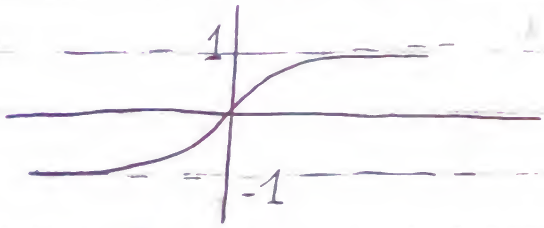
$$f(x) = \frac{1}{1+e^{-x}}$$



Bipolar Sigmoid

$$g(x) = \frac{2}{1+e^{-x}} - 1$$

$$g(x) = \frac{e^{-x} - 1}{e^{-x} + 1}$$



issues with Binary threshold:

1. abrupt change
2. not diffentiable
3. at zero, we cannot determine $f(x)$, some assume it to be 0.5, zero, 1
4. not continuous
5. function is not reversable, if you know $S = 1$, you cannot know the value of y since $0 < y < \infty$

Binary Sigmoidal function :- $f(x) = \dfrac{1}{1+e^{-x}}$

problem 1:

sketch the graph of the binary sigmoidal function

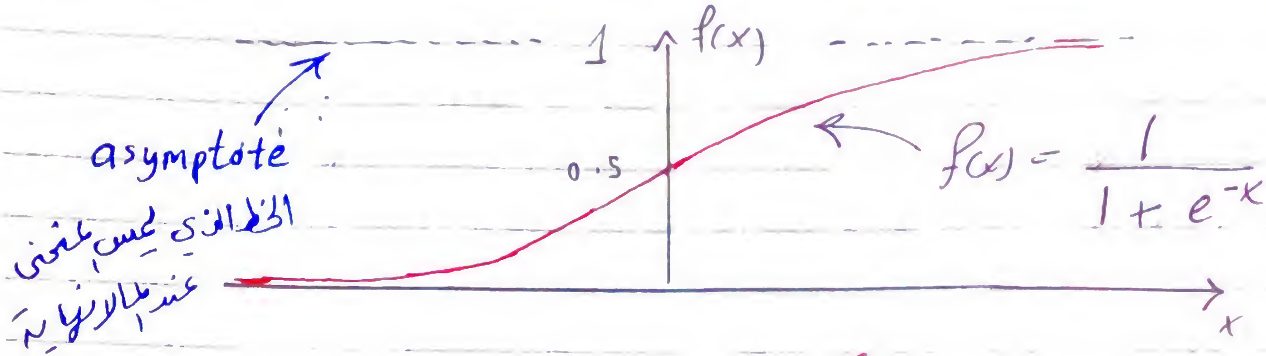
$$f(x) = \frac{1}{1+e^{-x}}$$

verify that $0 < f(x) < 1$

$f(-x) = 1 - f(x)$

calculate $f(0)$; $f(2)$; $f(-2)$

X	-5	-4	-3	-2	-1	0	1	2	3	4	5
f(x)	0.007	0.018				0.5				0.982	0.993



~~#~~ we have 2 asymptotes: ① $f(x) = 1$; $f(x) = -1$

to verify, we can sketch or substitute numirically

by me

$$f(0) = \frac{1}{1+e^0} = 1/2, \quad f(\infty) = \frac{1}{1-0} = 1$$

$$f(-\infty) = \frac{1}{\infty} = 0 \Rightarrow 0 < f(x) < 1$$

$$f(-4) = 0.018 = 1 - f(\overset{4}{x}) = 1 - 0.982 = 0.018$$

OR

$$1 - f(x) = 1 - \frac{1}{1+e^{-x}} = \frac{\cancel{1} + e^{-x} - \cancel{1}}{1+e^{-x}} = \frac{1}{e^x+1}$$

$$= \frac{1}{1+e^{-(-x)}} = f(-x)$$

* calculate ;

① $f(0) = \frac{1}{1+e^0} = \frac{1}{2}$

② $f(2) = \frac{1}{1+e^{-2}} = 0.881$

③ $f(-2) = 1 - 0.881 = 0.119$

$= \frac{1}{1+e^2} = 0.119$

Problem 2:-

Convince yourself that $f(x)$ is bounded, continuous, monotonically increasing, differentiable at all points.

1. bounded by 1 and 0
2. continuous $\longrightarrow$ Smooth continuous curve
3. monotonically increasing, $f(x)$ increases with increasing x
4. ~~we can~~ we can draw tangent line at any point

Problem 3:-

Verify that $x = \ln f(x) - \ln[1 - f(x)]$

$$= \ln\left[\frac{f(x)}{1 - f(x)}\right]$$

* بمعنى آخر: أثبت أنك تقدر معرفة X بمعرفة $f(x)$

أي أنها reversable

$$f(x) = \frac{1}{1 + e^{-x}}$$

$$f(x) + e^{-x} f(x) = 1$$

$$e^{-x} f(x) = 1 - f(x)$$

$$e^{-x} = \frac{1 - f(x)}{f(x)}$$

$$-x = \ln\left[\frac{1 - f(x)}{f(x)}\right]$$

$$x = \ln\left[\frac{f(x)}{1 - f(x)}\right]$$

$$= \ln f(x) - \ln[1 - f(x)]$$

* calculate x if $f(x) = 0.75$

$$\boxed{x = \ln\left[\frac{0.75}{1 - 0.75}\right] = 1.099}$$

Problem 4:

verify that

$$\frac{d\,f(x)}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} \longleftarrow (1)$$

يستخدم هذا القانون عندما لا تكون معروفة

OR alternatively ; $\frac{d\,f(x)}{dx} = f(x)\left[1 - f(x)\right]$ (2)

يستخدم هذا القانون عندما $f(x)$ معروفة .

$$f(x) = \frac{1}{1+e^{-x}}$$

$$\frac{d\,f(x)}{dx} = -\frac{e^{-x}}{(1+e^{-x})^2} \checkmark \quad (1)$$

ويمكن التعبير عن هذا التفاضل بالطريقة الآتية :

$$\frac{d\,f(x)}{d(x)} = \frac{1}{1+e^{-x}} \cdot \frac{e^{-x}}{1+e^{-x}} = \frac{1}{1+e^{-x}} \cdot \frac{1+e^{-x}-1}{1+e^{-x}}$$

$$= \frac{1}{1+e^{-x}}\left[1 - \frac{1}{1+e^{-x}}\right]$$

$$= f(x)\left[1 - f(x)\right]$$

Problem 5:- الشكل البياني للتفاضل كدالة في x

(1) sketch $\frac{d\,f(x)}{dx}$; prove that the graph is Symmetric about $x=0$ axis

(2) prove that $\frac{d\,f(x)}{dx}$ has a maximum value of 0.25 at $x=0$;

(3) calculate $\left.\frac{d\,f(x)}{dx}\right|_{x=2}$; $\left.\frac{d\,f(x)}{dx}\right|_{x=-2}$

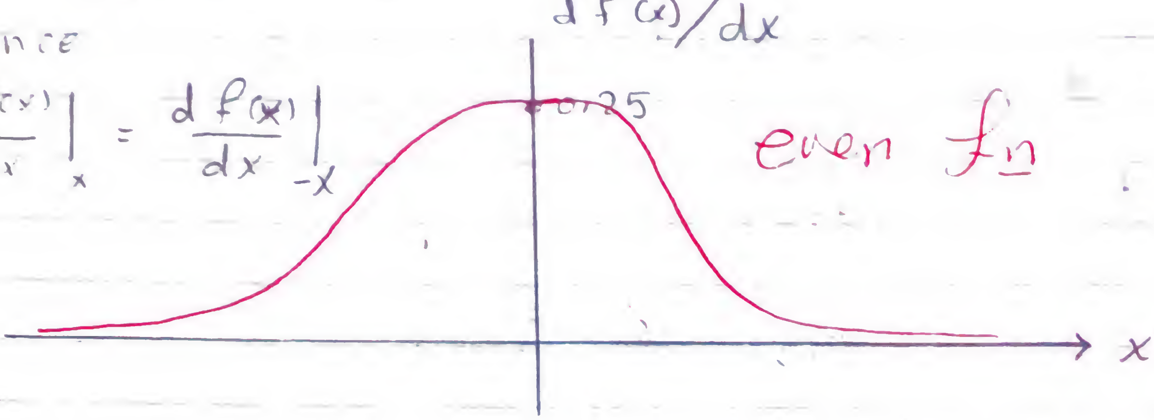
$$\frac{df(x)}{dx} = \frac{e^{-x}}{(1+e^{-x})^2}$$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\frac{df(x)}{dx}$	0.007	0.018				0.25				0.018	0.007

$\frac{df(x)}{dx}$ is even function $\rightarrow$ symmetric

since

$$\left.\frac{df(x)}{dx}\right|_{x} = \left.\frac{df(x)}{dx}\right|_{-x}$$



even fn

يتضح من الرسم ما يلي

① الدالة زوجية (even) حيث انها متماثلة حول المحور الرأسي

② هذا الرسم قيمة عظمى تساوي 0.25 وتحدث عندما $x=0$

$$\left.\frac{df(x)}{d(x)}\right|_{x=2} = 0.105 \quad ; \quad \left.\frac{df(x)}{d(x)}\right|_{x=-2} = 0.105.$$

Problem 6 -

for a given value of $\frac{df(x)}{dx}$ verify that there are two values of x; x_1 and x_2 such that

$$x_1 + x_2 = 0$$

$$f(x_1) + f(x_2) = 1$$

calculate $x_1, x_2, f(x_1), f(x_2)$ when $\frac{df(x)}{dx} = 0.15$

$f(x_2) = 1 - f(x_1)$ [from problem 1]

$f(x_1) + f(x_2) = 1$

يمكن إثبات النتائج السابقة بطريقة أخرى كمايلي:

from problem 4

$$\frac{d\,f(x)}{d(x)} = f(x)\left[1 - f(x)\right]$$

$$\frac{d\,f(x)}{d(x)} = f(x) - f^2_{(x)}$$

$$f^2(x) - f(x) + \frac{d\,f(x)}{dx} = 0$$

معادلة من الدرجة الثانية في x ، فيصبح هناك جذران للمعادلة

two roots

$f_1(x) = f(x_2)$; $f_2(x) = f(x_2)$

$$f_1(x) + f_2(x) = -\frac{b}{a} = -\frac{(-1)}{1} = 1$$

$\therefore\ f(x_1) + f(x_2) = 1$

$$* \quad \frac{1}{1+e^{-x_1}} + \frac{1}{1+e^{-x_2}} = 1$$

$$1 + e^{-x_2} + 1 + e^{-x_1} = (1+e^{-x_1})(1+e^{-x_2})$$

$$2 + \cancel{e^{-x_1}} + \cancel{e^{-x_2}} = 1 + \cancel{e^{-x_1}} + \cancel{e^{-x_2}} + e^{-(x_1+x_2)}$$

$$e^{-(x_2+x_1)} = 1 \qquad \ln$$

$$-x_2 - x_1 = \ln 1 = 0$$

$$x_1 + x_2 = 0$$

* given $\frac{d f(x)}{dx} = 0.15$

$$f^2(x) - f(x) + 0.15 = 0$$

$$f(x_1) = \frac{1 + \sqrt{1 - 4(0.15)}}{2} = 0.816$$

~~$f(x_2) =$~~

$$f(x_2) = \frac{1 - \sqrt{1-4(0.15)}}{2} = 0.184$$

$$= 1 - f(x_1)$$

$$x_1 = \ln\left[\frac{f(x_1)}{1 - f(x_1)}\right] \quad \text{given } f(x_1)$$

$$= \ln\left[\frac{0.816}{1-0.816}\right] = 1.489$$

$$x_2 = -x_1 = \ln\left[\frac{f(x_2)}{1 - f(x_2)}\right] = \ln\left[\frac{0.184}{1-0.184}\right]$$

$$= -1.489$$

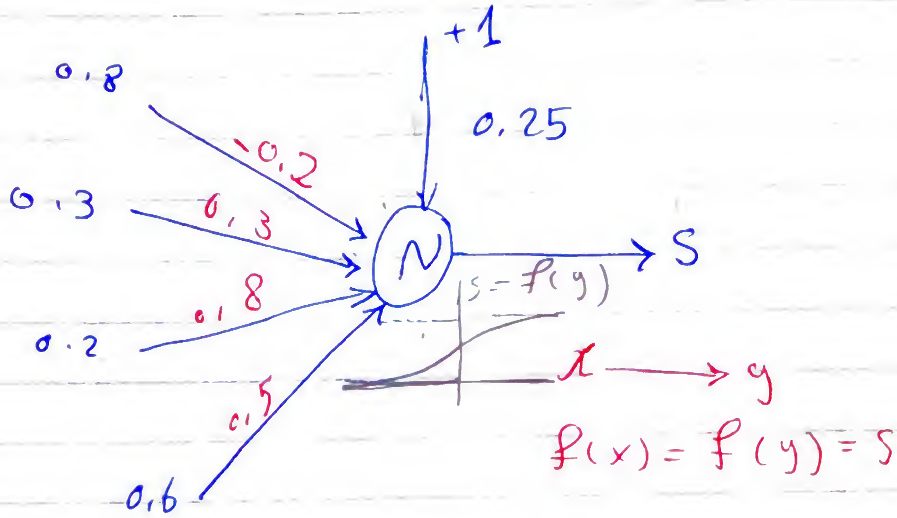
Problem 7:-

عندنا neuron بيستخدم Binary sigmoidal fn ويستخدم 4 input

$X_1 = 0.8$; $X_2 = 0.3$; $X_3 = 0.2$; $X_4 = 0.6$

$w_1 = -0.2$; $w_2 = 0.3$; $w_3 = 0.8$; $w_4 = 0.5$ بأوزان على الترتيب

احسب الخرج اذا كان Bias weight ب 0.25



$$S = f(y) = \frac{1}{1 + e^{-y}}$$

Activation (y)

$$y = (0.8)(-0.2) + (0.3)(0.3) + (0.2)(0.8) + (0.6)(0.5)$$
$$+ \underbrace{(1)(0.25)}_{\text{Bias}}$$
$$= 0.64$$

neuron output for a binary sigmoidal fn

$$S = \frac{1}{1 + e^{-y}} = \frac{1}{1 + e^{-0.64}} = \boxed{0.655}$$

Problem 8

عندنا neuron بيستخدم "BSF" (Binary Sigm. fn) ؛ ايه الـ activation (y) عندها

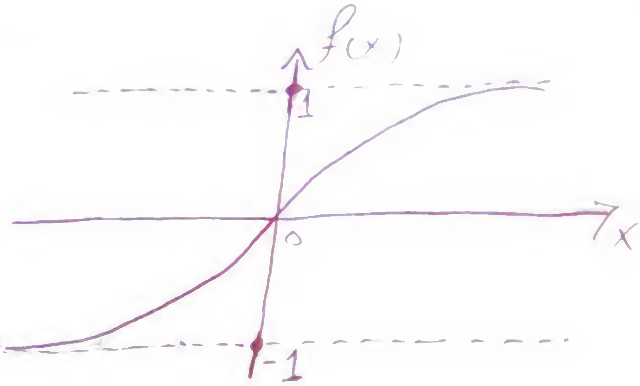
تكون الـ output (s) بـ 0.44

* هنا حط الرقم ومطلوب الدخل (عكس المسألة السابقة)

$$y = \ln\left[\frac{S}{1 - S}\right] \qquad [0 < S < 1] \text{ شرط}$$

$$y = \ln\left[\frac{0.44}{1 - 0.44}\right] = \boxed{-0.241}$$

Bipolar Sigmoidal $f_{\underline{n}}$



$$g(x) = \frac{2}{1+e^{-x}} - 1$$

$$= \frac{1-e^{-x}}{1+e^{-x}}$$

$$g(x) = \frac{2}{1+e^{-x}} - 1$$

$$= \frac{2-1-e^{-x}}{1+e^{-x}} = \frac{1-e^{-x}}{1+e^{-x}} \quad [\text{Proof}]$$

Problem 9

Sketch the bipolar sigmoidal fn.

$$g(x) = \frac{2}{1+e^{-x}} - 1$$

Verify that

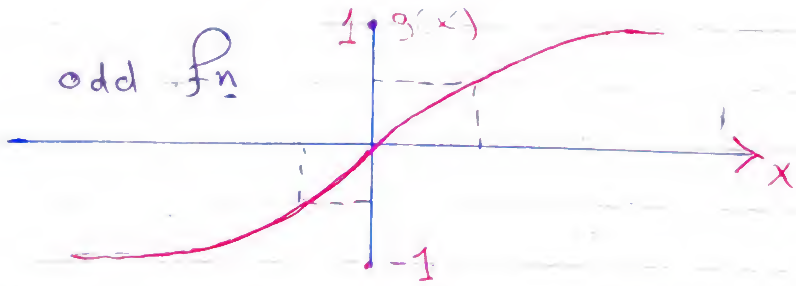
$$-1 < g(x) < 1$$

$$g(-x) = -g(x)$$

Calculate $g(0)$, $g(2)$, and $g(-2)$

X	-5	-4	-3	-2	-1	0	1	2	3	4	5
$g(x)$	-0.984	-0.444				0				0.964	0.986

odd $f_{\underline{n}}$



$g(-x) = -g(x)$ يمكن إثبات العلاقة

بالطريقة الآتية.

$$g(x) = \frac{2}{1+e^{-x}} - 1 \longrightarrow g(-x) = \frac{2}{1+e^{x}} - 1$$

$$g(-x) = \frac{2}{1+e^{x}} - 1 = \frac{2-1-e^{x}}{1+e^{x}}$$

$$= \frac{1-e^{x}}{1+e^{x}} = \frac{e^{-x}-1}{e^{-x}+1} = \frac{e^{-x}-2+1}{e^{-x}+1}$$

$$= \frac{-2}{1+e^{-x}} + 1 = -\left[\frac{2}{1+e^{-x}} - 1\right]$$

$$= -g(x)$$

$$g(0) = \frac{2}{1+e^{0}} - 1 = 1 - 1 = 0$$

$$g(2) = \frac{2}{1+e^{-2}} - 1 = 0.762$$

$$g(-2) = -g(2) = -0.762 = \frac{2}{1+e^{2}} - 1$$

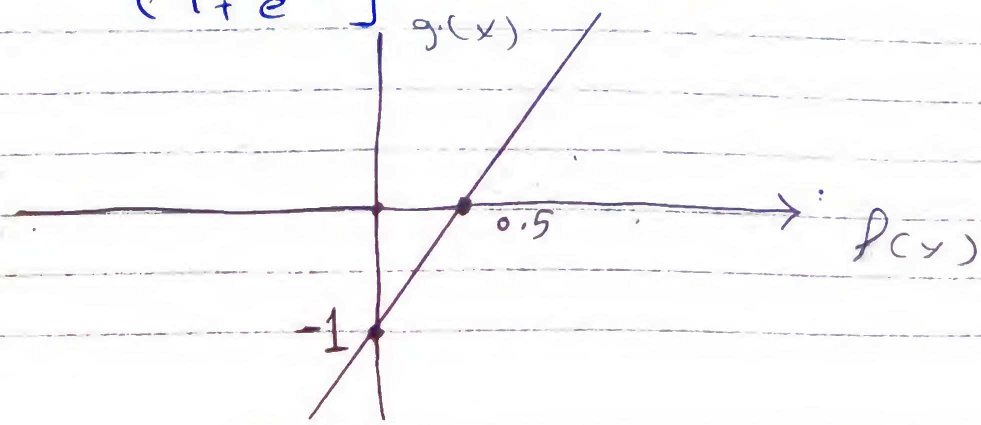
Problem 10

* Verify that $g(x) = 2f(x) - 1$

* Represent the relation grapically,
* What is the value of $g(x)$ when $f(x) = 0.5$?
* what is the value of $f(x)$ when $g(x)$ $g(x) = -1$

$$f(x) = \frac{1}{1+e^{-x}} \quad ; \quad g(x) = \frac{2}{1+e^{-x}} - 1$$

$$= 2\left[\frac{1}{1+e^{-x}}\right] - 1 = 2f(x) - 1$$



$f(x) = 0.5 \longrightarrow g(x) = 2(0.5) - 1 = 0$

$g(x) = -1 \longrightarrow f(x) = \frac{1-1}{2} = 0$

$f(x)$ عكس $g(x)$ لهما نفس منحنى

Problem 11

verify that

غير معرفة   غير معرفة

x $g(x)$

* $\frac{d\,g(x)}{dx} = \frac{2e^{-x}}{(1+e^{-x})^2} = 0.5\left[1 - g^2(x)\right]$

* $\frac{dg(x)}{dx} = 2\frac{d\,f(x)}{dx}$; sketch التفاضل وبين ملاحظاتك.

* calculate $\left.\frac{dg(x)}{dx}\right|_{x=2}$ and $\left.\frac{dg(x)}{dx}\right|_{x=-2}$

$$g(x) = \frac{2}{1+e^{-x}} - 1 = \frac{1-e^{-x}}{1+e^{-x}}$$

$$\frac{dg(x)}{dx} = \frac{(1+e^{-x})(e^{-x}) - (1-e^{-x})(-e^{-x})}{(1+e^{-x})^2}$$

$$= \frac{e^{-x} \cancel{+e^{-2x}} + e^{-x} \cancel{-e^{-2x}}}{(1+e^{-x})^2} = \frac{2e^{-x}}{(1+e^{-x})^2}$$

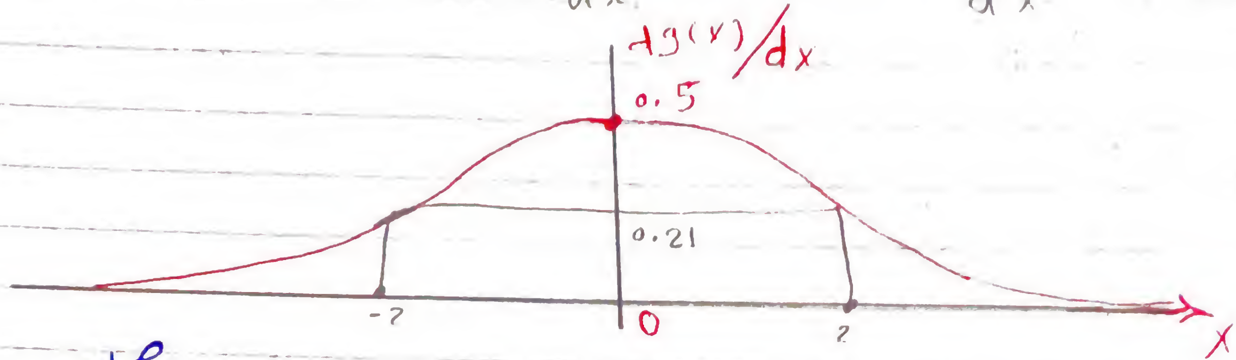
from problem 4 we conclude $\frac{dg(x)}{dx} = 2\frac{d\,f(x)}{dx}$

$$\frac{d\,g(x)}{dx} = \frac{2e^{-x}}{(1+e^{-x})^2} = 0.5\left[\frac{4e^{-x}}{(1+e^{-x})^2}\right]$$

$$= 0.5\left[\frac{(1+e^{-x})^2 - (1-e^{-x})^2}{(1+e^{-x})^2}\right] = 0.5\left[1-g^2(x)\right]$$

X	-5	-4	-3	-2	-1	0	1	2	3	4	5
$dg(x)/dx$	0.014	0.036				0.5				0.036	0.14

لاحظ ان قيم $\frac{dg(x)}{dx}$ هي ضعف قيم $\frac{df(x)}{dx}$ المناظرة في الجدول لحل مسألة (5)



* القيمة العظمى للتفاضل هي 0.5 عند $x=0$ شبيها بمنحنى $\frac{df(x)}{dx}$ في مسألة 5 ، ونجد $\frac{dg(x)}{dx}$ متماثل حول محور $x=0$

* قيمة $\frac{dg(x)}{dx}$ ، الى صفر عندما تؤول x ، الى $\infty, -\infty$

$$\left.\frac{dg(x)}{dx}\right|_{x=2} = \left.\frac{dg(x)}{dx}\right|_{x=-2} = 0.21$$

Problem 12

A Neuron has a bipolar sigmoidal function, Derive an expression for the activation y In terms of the neural outputs
Calculate y for $S=0.6$
and for y for $S=-0.6$

$$S = \frac{2}{1+e^{-y}} - 1 = \frac{1-e^{-y}}{1+e^{-y}}$$

$$S + Se^{-y} = 1 - e^{-y}$$

$$e^{-y} = \frac{1-S}{1+S}$$

$$-y = \ln\left[\frac{1-s}{1+s}\right]$$

$$y = \ln\left[\frac{1+s}{1-s}\right] = \ln(1+s) - \ln(1-s)$$

$$y = \ln\left[\frac{1+0.6}{1-0.6}\right] = 1.386 \qquad -1 < s < 1$$

$$s = -0.6 \longrightarrow s = \ln\left[\frac{1-0.6}{1+0.6}\right] = -1.386$$